Homework 6

1. **RSA Assumption (5+12+5).** Consider RSA encryption scheme with parameters $N = 35 = 5 \times 7$.

   (a) Compute $\varphi(N)$ and write down the set $\mathbb{Z}_N^*$.
   Solution.

   (b) Use repeated squaring and complete the rows X, X^2, X^4 for all $X \in \mathbb{Z}_N^*$ as you have seen in the class (slides), that is, fill in the following table by adding as many columns as needed.
   Solution.

X	1	2	3	4	6	8	9	11	12	13	16	17
X^2												
X^4												

X	18	19	22	23	24	26	27	29	31	32	33	34
X^2												
X^4												

(c) Find the row X^5 and show that X^5 is a bijection from $\mathbb{Z}_N^*$ to $\mathbb{Z}_N^*$.
Solution.

X	1	2	3	4	6	8	9	11	12	13	16	17
X^4												
X^5												

X	18	19	22	23	24	26	27	29	31	32	33	34
X^4												
X^5												

2. **Answer the following questions (7+7+7+7 points):**

   (a) (7 points) By hand, compute the three least significant (decimal) digits of $6251007^{1960404}$. Explain your logic.

   Solution.

(b) (7 points) Is the following RSA signature scheme valid? (Justify your answer)

$$(r\|m) = 24, \sigma = 196, N = 1165, e = 43$$

Here, m denotes the message, r denotes the randomness used to sign m, and σ denotes the signature. Moreover, $(r\|m)$ denotes the concatenation of r and m. The signature algorithm $Sign(m)$ returns $(r\|m)^d \mod N$ where d is the inverse of e modulo $\varphi(N)$. The verification algorithm $Ver(m, \sigma)$ returns $((r\|m) == \sigma^e \mod N)$.

Solution.

(c) (7 points) Remember that in RSA encryption and signature schemes, $N = p \times q$ where p and q are two large primes. Show that in the RSA scheme (with public parameters N and e), if you know N and $\varphi(N)$, then you can efficiently factorize N, i.e., you can recover p and q.

Solution.

(d) (7 points) Consider an encryption scheme where $Enc(m) := m^e \mod N$ where e is a positive integer relatively prime to $\varphi(N)$ and $Dec(c) := c^d \mod N$ where d is the inverse of e modulo $\varphi(N)$. Show that in this encryption scheme, if you know the encryption of m_1 and the encryption of m_2, then you can find the encryption of $(m_1 \times m_2)^5$.

Solution.

(e) (7 points) Suppose $n = 11413 = 101 \cdot 113$, where 101 and 113 are primes. Let $e_1 = 8765$ and $e_2 = 7653$.

i. (2 points) Only one of the two exponents e_1, e_2 is a valid RSA encryption key, which one?
Solution.

ii. (3 points) For the valid encryption key, compute the corresponding decryption key d.
Solution.

iii. (2 points) Decrypt the cipher text $c = 3233$.
Solution.

3. **Euler Phi Function (30 points)**

   (a) (10 points) Let $N = p_1^{e_1} \cdot p_2^{e_2} \cdots p_t^{e_t}$ represent the unique prime factorization of a natural number N, where $p_1 < p_2 < \cdots < p_t$ are prime numbers and $e_1, e_2, \dots, e_t$ are natural numbers. Let $\mathbb{Z}_N^* = \left\{x \colon 0 \leqslant x < N-1, \gcd(x, N) = 1\right\}$ and $\varphi(N) = \left|\mathbb{Z}_N^*\right|$. Using the inclusion exclusion principle, prove that

$$\varphi(N) = N \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_t}\right).$$

   Solution.

(b) (5 points) For any $x \in \mathbb{Z}_N^*$, prove that

$$x^{\varphi(N)} = 1 \mod N.$$

Hint: Consider the subgroup generated by x and its order.

Solution.

(c) **Replacing $\varphi(N)$ with $\frac{\varphi(N)}{2}$ in RSA.** (15 points)

In RSA, we pick the exponent e and the decryption key d from the set $\mathbb{Z}^*_{\varphi(N)}$. This problem shall show that we can choose $e, d \in \mathbb{Z}^*_{\varphi(N)/2}$ instead.

Let p, q be two distinct odd primes and define $N = pq$.

i. (2 points) For any $e \in \mathbb{Z}^*_{\varphi(N)/2}$, prove that $x^e \colon \mathbb{Z}^*_N \to \mathbb{Z}^*_N$ is a bijection.
Solution.

ii. (7 points) Consider any $x \in \mathbb{Z}^*_N$. Prove that $x^{\frac{\varphi(N)}{2}} = 1 \mod p$ and $x^{\frac{\varphi(N)}{2}} = 1 \mod q$.
Solution.

iii. (3 points) Consider any $x \in \mathbb{Z}_N^*$. Prove that $x^{\frac{\varphi(N)}{2}} = 1 \mod N$.
Solution.

iv. (3 points) Suppose e, d are integers that $e \cdot d = 1 \mod \frac{\varphi(N)}{2}$. Show that $(x^e)^d = x \mod N$, for any $x \in \mathbb{Z}_N^*$.
Solution.

4. **Understanding hardness of the Discrete Logarithm Problem.** (15 points) Suppose $(G, \circ)$ is a group of order N generated by $g \in G$. Suppose there is an algorithm $\mathcal{A}_{DL}$ that, when given input $X \in G$, it outputs $x \in \{0, 1, \dots, N-1\}$ such that $g^x = X$ with probability p_X.

   Think of it this way: The algorithm $\mathcal{A}_{DL}$ solves the discrete logarithm problem; however, for different inputs $X \in G$, its success probability p_X may be different.

   Let $p = \frac{\left(\sum_{X \in G} p_X\right)}{N}$ represent the average success probability of $\mathcal{A}_{DL}$ solving the discrete logarithm problem when X is chosen uniformly at random from G.

   Construct a new algorithm $\mathcal{B}$ that takes *any* $X \in G$ as input and outputs $x \in \{0, 1, \dots, N-1\}$ (by making one call to the algorithm $\mathcal{A}_{DL}$) such that $g^x = X$ with probability p. This new algorithm that you construct shall solve the discrete logarithm problem for *every* $X \in G$ with the same probability p.

   (*Remark:* Intuitively, this result shows that solving the discrete logarithm problem for *any* $X \in G$ is no harder than solving the discrete logarithm problem for a *random* $X \in G$.)

   Solution.

5. **Concatenating a random bit string before a message.** (15 points)

Let $m \in \{0,1\}^a$ be an arbitrary message. Define the set

$$S_m = \left\{ (r \| m) \colon r \in \{0,1\}^b \right\}.$$

Let p be an odd prime. Recall that in the RSA encryption algorithm, we encrypted a message y chosen uniformly at random from this set S_m.

Prove the following

$$\Pr_{y \xleftarrow{\$} S_m} [p \text{ divides } y] \leqslant 2^{-b} \cdot \left\lceil 2^b/p \right\rceil.$$

(*Remark:* This bound is tight as well. There exists m such that equality is achieved in the probability expression above. Intuitively, this result shows that the message y will be relatively prime to p with probability (roughly) $(1 - 1/p)$.)

Solution.

6. **Properties of x^e when e is relatively prime to $\varphi(N)$** (20 points)

   In this problem, we will partially prove a result from the class that was left unproven. Suppose $N = pq$, where p and q are distinct prime numbers. Let e be a natural number that is relatively prime to $\varphi(N) = (p-1)(q-1)$. In the lectures, we claimed (without proof) that the function $x^e\colon \mathbb{Z}_N^* \to \mathbb{Z}_N^*$ is a bijection. The following problem is key to proving this result.

Let $N = pq$, where p and q are distinct prime numbers. Let e be a natural number relatively prime to $(p-1)(q-1)$. Consider $x, y \in \mathbb{Z}_N^*$. If $x^e = y^e \mod N$, then prove that $x = y$.

   Hint: You might find the following facts useful.

   - Every $\alpha \in \mathbb{Z}_N$ can be uniquely written as (α_p, α_q) such that $\alpha = \alpha_p \mod p$ and $\alpha = \alpha_q \mod q$, using the Chinese Remainder theorem. We will write this observation succinctly as $\alpha = (\alpha_p, \alpha_q) \mod (p, q)$.
   - For $\alpha, \beta \in \mathbb{Z}_N$, and $e \in \mathbb{N}$ we have $\alpha^e = \beta \mod N$ if and only if $\alpha_p^e = \beta_p \mod p$ and $\alpha_q^e = \beta_q \mod q$. We will write this succinctly as $\alpha^e = (\alpha_p^e, \alpha_q^e) \mod (p, q)$.
   - From the Extended GCD algorithm, if u and v are relatively prime then, there exists integers $a, b \in \mathbb{Z}$ such that $au + bv = 1$.
   - Fermat's little theorem states that $x^{p-1} = 1 \mod p$ if x is a natural number that is relatively prime to the prime p.

   Solution.

7. **Challenging: Inverting exponentiation function.** (20 points)

   Fix $N = pq$, where p and q are distinct odd primes. Let e be a natural number such that $\gcd(e, \varphi(N)) = 1$. Suppose there is an adversary $\mathcal{A}$ running in time T such that

   $$\Pr\left[[\mathcal{A}([x^e \mod N]) = x]\right] = 0.01$$

   for x chosen uniformly at random from $\mathbb{Z}_N^*$. Intuitively, this algorithm successfully finds the e-th root with probability 0.01, for a random x.

   For any $\varepsilon \in (0,1)$, construct an adversary $\mathcal{B}_\varepsilon$ (which, possibly, makes multiple calls to the adversary $\mathcal{A}$) such that

   $$\Pr\left[[\mathcal{B}_\varepsilon([x^e \mod N]) = x]\right] = 1 - \varepsilon,$$

   for *every* $x \in \mathbb{Z}_N^*$. The algorithm $\mathcal{B}_\varepsilon$ should have a running time polynomial in $T, \log N$, and $\log 1/\varepsilon$.

   Solution.

Collaborators :